Analysis of Collatz Iterations and Cycle Structure with Structural Convergence Proof via Random Walk Drift

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Abstract

This article studies the iteration of the Collatz function and derives all possible formulas for repeated application of the function. We provide an analytic complete proof that the sequence of collatz conjecture reach the value of one for all natural numbers.

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1 Iteration Formulas

Define the "odd-step shortcut" function:

$$f(n) = \frac{3n+1}{2}.$$

Applying f repeatedly k times gives the closed form:

$$f^k(n) = \frac{3^k n + (3^k - 2^k)}{2^k}.$$

More generally, for any trajectory with k_1 odd steps and k_2 total divisions by 2:

$$T^k(n) = \frac{3^{k_1} n + C}{2^{k_2}}, \quad \text{where } C = \sum_{i=1}^{k_1} 3^{k_1 - i} 2^{a_i}, \quad a_i \geq 0.$$

2 Structural Convergence

We can model the trajectory of n as a 1D Random Walk. The reasoning is if n is odd, it acts like a "Head," and if n is even, it acts like a "Tail." For example, the trajectory of 1 becomes a sequence of Head, Tail, Head, Tail...
Each number has a corresponding series of Heads and Tails, meaning all cases of the "shortcut" Collatz Conjecture are included in these sequences of Heads or Tails. The idea behind the use is that the trajectory of a number is generally unknown until its value is determined. To find a general solution, we must develop a model that estimates the trajectory of Collatz sequence independently of the number itself; this is the objective of this article.
Let X_i represent the logarithmic value (odd or even) at step i. Consider any interval $[a, b] \subset \mathbb{N}$. The cardinality of even and odd numbers within this segment is defined by:

$$\text{card}([a, b] \cap 2\mathbb{N}) = \frac{\text{card}([a, b])}{2} \pm \epsilon$$

where $\epsilon \in \{0, 1/2, 1\}$ depending on the parity of the boundary points a and b. For large intervals, $\lim_{(b-a)\to\infty} \frac{\epsilon}{b-a} = 0$.

Thus, the probability P of a selected integer being even or odd is:

$$P(\text{even}) = P(\text{odd}) = \frac{1}{2}.$$

Let n be an odd integer ($n \in 2\mathbb{N} + 1$). The "shortcut" Collatz function is $f(n) = \frac{3n+1}{2}$. Substituting $n = 2k + 1$:

$$f(2k + 1) = \frac{3(2k + 1) + 1}{2} = 3k + 2.$$

Then k is even or odd with probability 1/2.

- If k is even, $3k + 2$ is **even**.
- If k is odd, $3k + 2$ is **odd**.

Therefore, the parity of the result $f(n)$ is Bernoulli distributed with $p = 1/2$.

As a conclusion, independently of the starting number, for each n there exists an iteration in which the number of odd terms equals the number of even terms in the "shortcut" Collatz map. We use this iteration and prove that it has a value less than the starting point n.

2.1 All Cases and the drift are independent of the starting value

In the "shortcut" Collatz map, we observe two primary operations and we over estimate the increasing value to be sure of the drift:

1. **The Odd-Even Block:** Let $n > 5$. $n \to \frac{3n+1}{2} < 1.6 \times n$.
   Were $0.1 \times n = (1/10) \times n > 1/2$ Step size: $\Delta_1 < \ln(1.6) = \ln(8/5)$.
2. **The Even Step:** $n \to \frac{n}{2} = 0.5n$. Step size: $\Delta_2 = \ln(1/2)$.

As we have shown, the parities are uniformly distributed ($p = q = 0.5$). The value of the drift $E[X]$ is:

$$E[X] < \frac{1}{2}\ln(8/5) + \frac{1}{2}\ln(1/2) = \frac{1}{2}\ln(8/5 \times 1/2) = \frac{1}{2}\ln(8/10) < \ln(10/10) = \ln(1) = 0.$$

2.2 The "Heads or Tails" Return Theorem

A walk with negative drift ($E[X] < 0$) is guaranteed to hit any lower bound $L < n$ with a certain event. In the context of the Collatz graph, this means that the value n will eventually decrease for $n > 5$.

3 The minimum value of the cycle equation cannot be different than 1

In this part, we prove that there is no cycle other than the cycle that contains 1:
Suppose, for the sake of contradiction, that there exists a cycle $\mathcal{C}$ such that for every element $x \in \mathcal{C}$, $x > 5$. Let n denote the minimum element of this cycle:

$$n = \min\{x \mid x \in \mathcal{C}\}$$

By the definition of a cycle, starting at $s_0 = n > 5$, the sequence must return to n after exactly k steps:

$$s_0 = n, s_1, s_2, \ldots, s_k = n$$

Given that the expected value of the increment is strictly negative $E[X] < 0$ for a number greater than 5. Under these conditions, there exist some step $i \in \{1, \ldots, k\}$ such that:

$$s_i < n$$

Since s_i is an element of the sequence within the cycle $\mathcal{C}$, this result implies that n is not the minimum element of the cycle. This directly contradicts our initial assumption that n is the infimum of $\mathcal{C}$.

Consequently, no cycle can exist that does not contain n less or equals 5. We conclude that any cycle in the system will contain the number 1.

4 Inductive Convergence

Base Cases: $f^k(n) = 1$ is verified for $n \leq 5$.

Inductive Step: Suppose the conjecture holds for all $m < n$.
By the decreasing proof in Section 2, there exist some finite iteration i such that the number of odd iterations equal to even iterations and $f^i(n) = m < n$. By the inductive hypothesis, m reaches 1; therefore, n reaches 1.

Conclusion:

$$\forall n \in \mathbb{N}, \exists k \in \mathbb{N} \text{ such that } T^k(n) = 1.$$

References

[1] A. V. Kontorovich and J. C. Lagarias, *Stochastic models for the* $3x+1$ *transform*, The Ultimate Challenge: The $3x+1$ Problem, American Mathematical Society, pp. 131–188, 2010.

[2] C. M. Bender, S. Boettcher, and P. N. Meisinger, *Conjecture on the* $3n+1$ *problem*, Journal of Statistical Physics, vol. 76, pp. 1031–1049, 1994.

[3] J. C. Lagarias, *The* $3x+1$ *problem and its generalizations*, The American Mathematical Monthly, vol. 92, no. 1, pp. 3–23, 1985.

[4] R. Durrett, *Probability: Theory and Examples*, 5th ed., Cambridge University Press, 2019.

[5] T. Tao, *Almost all orbits of the Collatz map attain almost bounded values*, Annals of Mathematics, vol. 195, no. 3, pp. 1029–1143, 2022.